Velocity correlations for turbulent shear flow

1984 J. Phys. A: Math. Gen. 17 L555

(http://iopscience.iop.org/0305-4470/17/10/007)

J. Phys. A: Math. Gen. 17 (1984) L555–L558. Printed in Great Britain

LETTER TO THE EDITOR

Velocity correlations for turbulent shear flow

M J Ortiz and A Ruiz de Elvira
Departamento de Física, Universidad de Alcalá, Campus, Alcalá de Henares, Madrid, Spain

Received 28 March 1984

Abstract. Spectral functions corresponding to velocity correlations are derived for fully developed turbulent shear flow in a plane channel. These functions follow rather simple differential equations. It is found that, for high shear, terms with different decay rates in orthogonal directions are needed.

A profitable approach to the study of turbulent homogeneous flows is to analyse the effects that a weak mean strain produces on an existing homogeneous isotropic turbulent field. It is well known that, for small values of the shear, one of these effects is a stretching of the vortices in the direction of the mean flow. It has been pointed out by Leslie (1973) that a formal expansion for the correlations between components of the fluctuating velocities in wavenumber space is possible:

$$q_{im}(k, t) = q_{im}^{(0)}(\boldsymbol{k}, t) + q_{im}^{(1)}(\boldsymbol{k}, t) \tag{1}$$

where $q_{im}^{(0)}$ is the correlation tensor of the background isotropic turbulent field, and $q_{im}^{(1)}$ is a first-order correction corresponding to the correlation between this background field and the fluctuations induced by the mean strain. This expansion can be considered as a division of q_{im} into terms with different angular dependences. Thus, $q_{im}^{(0)}$ can be written as the product of a geometrical isotropic tensor times a function of the modulus of $\boldsymbol{k}$, and likewise, $q_{im}^{(1)}$ as an anisotropic geometrical tensor times another different function of the modulus of $\boldsymbol{k}$. This anisotropic factor must then account for all the effects of anisotropy present in the flow.

For real flows, Leslie (1970) suggested an expansion formally analogous to (1). It has been explicitly formulated for a plane channel but this formulation can be extended to other real flows. In the following x_1 is the streamwise coordinate while x_2 is the coordinate in the direction normal to the channel walls. In order to obviate the difficulty of going to the wavenumber space, we introduce, following Leslie, the transformation:

$$q_{im}(\boldsymbol{k}, t) = \frac{1}{(2\pi)^3} \int \mathrm{d}(\boldsymbol{x}-\boldsymbol{x}') q_{im}(\boldsymbol{x}-\boldsymbol{x}', Y) \exp[-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')] \tag{2}$$

where $Y = \frac{1}{2}(x_2 + x_2')$ is the x_2 centroid coordinate of $\boldsymbol{x}$ and $\boldsymbol{x}'$. For this spectrum, the proposed expansion reads

$$q_{im}(\boldsymbol{k}, Y) = \frac{l(Y)}{4\pi k^2}[E_0(Y)P_{im}(\boldsymbol{k})f^{(0)}(l(Y)k) + \tau(Y)V_{im}(\boldsymbol{k})f^{(1)}(l(Y)k)] \tag{3}$$

where $l(Y)$ is a correlation length, characteristic for each flow. The angular factor $P_{im}(\boldsymbol{k})$ has the form of a general isotropic tensor in k-space, while $V_{im}(\boldsymbol{k})$ is an antisymmetric tensor, except for $V_{12}=V_{21}$; in this way it does not contribute to the total energy of the field, but does to the Reynolds stress. The term $E_0(Y)$ is the turbulent energy at the point Y. $f^{(0)}$ and $f^{(1)}$ are spectral functions, supposed to be universal, i.e. valid for all kinds of real turbulent flows.

The convenience of performing an explicit calculation of the spectrum associated to the correlations in the inertial layer of a turbulent plane channel flow was suggested by Leslie (1973). This calculation would allow one to test the assumption of self-similarity for the symmetric part of the spectrum and to determine a characteristic length in this region. Besides, if it were possible to find an expansion of the form of (1), it would yield the form of functions $f^{(0)}$ and $f^{(1)}$. Such a calculation has been reported in Ortiz and Ruiz de Elvira (1984), in the following referred to as (I). We started with the equations for the correlations obtained using the DI approximation given by Kraichnan (1964). As these include integro-differential operators and a diagonalisation has not been found possible in our case, we tried some approximations, based on the characteristics of the flow. Among these are a reduction of the problem to two dimensions, a localisation of the pressure operator and an iterative procedure for solving the resulting equations. The validity of these approximations is checked by the good agreement between our calculated correlations and the available experimental data. The results presented in (I) seem conclusive with respect to the self-similarity of the spectrum associated to the symmetric part of the correlations. The functional form of the length scale $l(Y)$ obtained is in excellent agreement with Prandtl's mixing length. In (I) we also presented some results for the spectral functions.

The aim of this letter is to report recent and more detailed calculations which give us a new insight into the analysis of the spectrum, in particular the effects of a non-small shear on the vortices.

We are looking for functions of the modulus of the wavevector $\boldsymbol{k}$ or, equivalently, of the modulus of $(\boldsymbol{x}-\boldsymbol{x}')$. Therefore we try writing our calculated correlations as the sum of two terms. The first one, $\tilde{\boldsymbol{a}}^1$, is invariant under an interchange of the two variables (x_1-x_1') and (x_2-x_2'). The second, $\tilde{\boldsymbol{a}}^2$, takes into account the big difference in the decay rates of the correlations found in (I) for the two orthogonal directions x_1 and x_2:

$$\tilde{q}_{ij}(\boldsymbol{x}-\boldsymbol{x}', Y)=I(Y)\{\tilde{a}^1_{ij}(\boldsymbol{x}-\boldsymbol{x}', Y)+\tilde{a}^2_{ij}[\alpha(x_1-x_1'), (x_2-x_2'), Y]\}. \tag{4}$$

The above mentioned difference in the decay rates is represented by the factor α.

In a similar way, the term $\tilde{a}^1$ can be split into an even function of each of the coordinates (x_i-x_i'), plus an odd function of the same set of coordinates:

$$\tilde{a}^{(1)}_{ij}(\boldsymbol{x}-\boldsymbol{x}', Y)=\tilde{a}^{(1e)}_{ij}(\boldsymbol{x}-\boldsymbol{x}', Y)+\tilde{a}^{(1o)}_{ij}(\boldsymbol{x}-\boldsymbol{x}', Y). \tag{5}$$

If we Fourier transform (4) using definition (2), we obtain:

$$q_{ij}(\boldsymbol{k}, Y)=I(Y)[a^{(1e)}_{ij}(k, Y)+a^{(1o)}_{ij}(\boldsymbol{k}, Y)+a^{(2)}(k_1/\alpha, k_2, Y)]. \tag{6}$$

We have approximated the functions $a^{(1e)}_{ij}$ and $a^{(1o)}_{ij}$ by combinations of analytical functions. We only show here the result of these approximations for the case $i=j=2$:

$$a^{(1e)}_{22}(k, Y)\simeq 0.39 l^{2.5}(Y)\sqrt{k}K_{1/4}[(0.61/2)(l(Y)k)^2] \tag{7}$$

where $k=(k_1^2+k_2^2)^{1/2}$. $l(Y)=l(0.45\,Y^{0.915})$mm is the length scale in the logarithmic layer as determined in (I). $K_{1/4}$ is the modified Bessel function of the second kind.

For the term $a_{22}^{(1o)}$ we have found

$$a_{22}^{(1o)}(k, Y) \simeq (\pi\beta/2\sqrt{3})[1-2\beta(2k_1^2+2k_2^2+5k_1k_2)]\exp[-\beta(k_1^2+k_2^2+k_1k_2)] \tag{8}$$

where $\beta = l^2(Y)/\sqrt{3}$.

It is interesting to note that the third term $a_{ij}^{(2)}(k_1/\alpha, k_2, Y)$ is of the same order of magnitude as $a_{ij}^{(12)}$ and furthermore it is of qualitative importance for the dynamics of correlations. Therefore it cannot be dropped, although it cannot be written out as a simple combination of analytical functions.

Function $a_{22}^{(1e)}$ was obtained by Edwards and McComb (1971) as the solution of an equation related to the energy spectrum in the homogeneous isotropic turbulence. In their work they started by substituting the equation for transport in a turbulent flow by a local Fokker–Planck type equation. After scaling conveniently and discarding terms of high order in the wavenumber, they obtained for a function $A(k)$, related to the energy spectrum $E(k)$, the equation

$$(\mathrm{d}^2/\mathrm{d}k^2-\gamma^2k^2)A(k)=0 \tag{9}$$

which has as a solution

$$A(k)=\sqrt{k}K_{1/4}[(\gamma/2)k^2] \tag{10}$$

similar to (7). In figure 1 we present the plot of our calculated $a_{22}^{(1e)}(k, Y)$ for $Y=4$ mm and function $A(k)$ with $\gamma=0.61l^2(Y)$.

Function $a_{22}^{(1o)}$ follows the equation

$$[\nabla^2+L_1+20\beta-\beta^2(5k_1^2+5k_2^2+8k_1k_2)]a_{22}^{(1o)}(k_1, k_2, Y)=0 \tag{11}$$

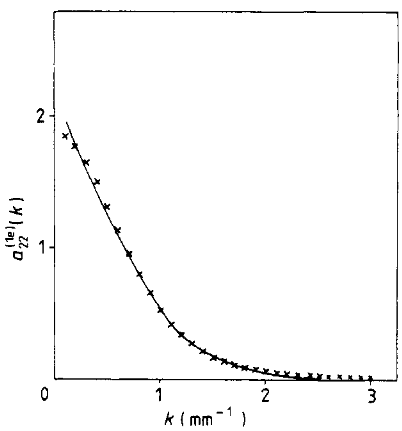

Figure 1. Even spectral function $a_{22}^{(1e)}(k)_{Y=4\mathrm{mm}}$, in arbitrary units. × denotes results of calculation reported in (I). The curve represents the function $\sqrt{k}K_{1/4}[(\alpha/2)k^2]$.

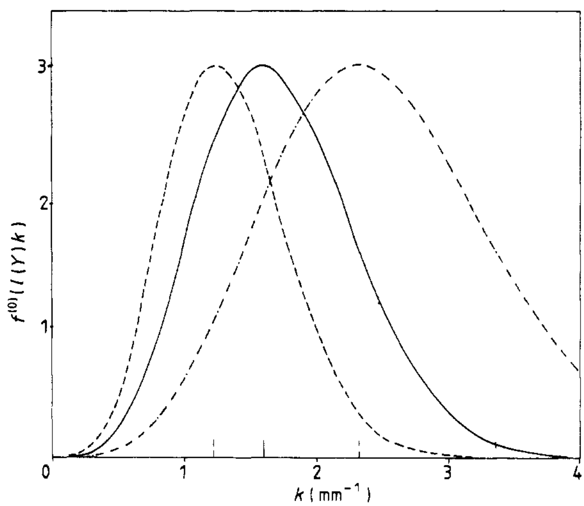

Figure 2. Function $f^{(0)}(l(Y)k)$ for three different values of centroid coordinate Y, in arbitrary units. All three curves are normalised to the same maximum value. Slashes in the curve for $Y=3$ mm denote the region for which $f^{(0)} \simeq k^{-5/3}$. Chain curve, $Y=2$ mm; full curve, $Y=3$ mm; broken curve, $Y=4$ mm.

where the operator L_1 is given by the following combination of first-order derivatives:

$$L_1 = k_1 k_2(\Sigma - \Delta) + \tfrac{1}{8}(\Sigma + 21\Delta)$$
$$\Sigma = (\partial_{k_1} + \partial_{k_2})/(k_1 + k_2), \qquad \Delta = (\partial_{k_1} - \partial_{k_2})/(k_1 - k_2). \tag{12}$$

Although (11) is a partial differential equation, its structure is formally analogous to that of the Bessel ordinary differential equation, and therefore it is related to (10).

At this point, to compare our expressions with Leslie's proposal (3) we must set apart from $a_{22}^{(1e)}$ an isotropic term and similarly, from $a_{22}^{(1o)}$ an anisotropic one. This presents some difficulty since the general expression for $P_{ij}(\boldsymbol{k})$ is obtained using the continuity equation in three dimensions. It is not feasible to take $k_3 = 0$, since this is equivalent to supposing $(x_3 - x_3') = \infty$. We therefore worked in configuration space and divided $a_{ij}^{(1e)}(\boldsymbol{x} - \boldsymbol{x}', Y)$ at $(x_3 - x_3') = 0$ into an isotropic tensor and a residual term. The Fourier transfom of the isotropic tensor has the form given in (3) with function $f^{(0)}(l(Y)k)$ given by

$$f^{(0)}(l(Y)k) = (l(Y)k)^{9/2} K_{1/4}[0.97(l(Y)k)^2] \tag{13}$$

and

$$E_0(Y) = 0.047\, Y^{3.2}/(l(Y))^{9/2}. \tag{14}$$

The fitting of $E_0(Y)$ to our results was done with determination coefficient $r^2 = 0.997$. In figure 2 we have plotted $f^{(0)}$ against its argument. For intermediate values of wavenumber k we can identify a range where the function behaves like $k^{-5/3}$.

Term $a_{ij}^{(1o)}$ can likewise be split into an anisotropic tensor and a residual term. For the anisotropic part we have identified the function $f^{(1)}(l(Y)k)$:

$$f^{(1)}(l(Y)k) = (l(Y)k)^6 \exp[-\varepsilon(l(Y)k)^2], \qquad \varepsilon = 1.66 \tag{15}$$

which for intermediate values of the wavenumber behaves like $k^{-7/3}$.

The functional forms found for $f^{(0)}$ and $f^{(1)}$ seem encouraging. Nevertheless, the spectrum contains more terms than is implied by equation (3): term $a_{ij}^{(2)}$ and the two residual terms produced by the division into isotropic and anisotropic parts of $a_{ij}^{(1e)}$ and $a_{ij}^{(1o)}$. These terms are not only not small but also qualitatively important. Our calculation seems conclusive in the sense that for high shear it is not enough to express its effect by the second term in (3). Other terms, accounting for the different decay rates along orthogonal axes, are needed. On the other hand, our results support the universality of functions $f^{(0)}$ and $f^{(1)}$.

As the identification of the factors $P_{ij}(\boldsymbol{k})$ and $V_{ij}(\boldsymbol{k})$ is problematic in the two-dimensional case, it is important to extend our calculations to three dimensions before analysing these additional terms. A preliminary study is under way.

We wish to thank Professor D C Leslie for stimulating correspondence.

References

Edwards S F and McComb W D 1971 *Proc. R. Soc.* A **325** 313
Kraichnan R H 1964 *Phys. Fluids* **7** 1048
Leslie D C 1970 *J. Phys. A: Gen. Phys.* **3** L16
—— 1973 *Developments in the Theory of Turbulence* (Oxford: Clarendon)
Ortiz M J and Ruiz de Elvira A 1984 *J. Mecanique* **3** (1) 63